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Instanton-Meron Hybrid in the Background of Gravitational Instantons

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Abstract

When it comes to the topological aspect, gravity may have profound effects even at the level of particle physics despite its negligibly small relative strength well below the Planck scale. In spite of this intriguing possibility, relatively little attempt has been made toward the exhibition of this phenomenon in relevant physical systems. In the present work, perhaps the simplest and the most straightforward new algorithm for generating solutions to (anti) self-dual Yang-Mills (YM) equation in the typical gravitational instanton backgrounds is proposed and then applied to find the solutions practically in all the gravitational instantons known. Solutions thus obtained turn out to be some kind of instanton-meron hybrids possessing mixed features of both. Namely, they are rather exotic type of configurations obeyng first order (anti) self-dual YM equation which are everywhere non-singular and have finite Euclidean YM actions on one hand while exhibiting meron-like large distance behavior and carrying generally *fractional* topological charge values on the other. Close

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inspection, however, reveals that the solutions are more like instantons rather than merons in their generic natures.

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I. Introduction

Certainly the discovery of the topologically degenerate vacuum structure of non-abelian gauge theories was the starting point from which we began to appreciate the fruitful but still mysterious non-perturbative regime of the theories. And central to this non-perturbative aspects of non-abelian gauge theories is the pseudoparticles, dubbed "instantons" [1]. In a naive mathmatical sense, they are the classical solutions to Euclidean field equations of non-abelian gauge theories and in a physical sense, they are the non-abelian gauge field configurations interpolating between two homotopically distinct but degenerate vacua. They thus can be thought of as saddle points which make dominant contribution to the intervacua tunnelling amplitude in the path integral formulation of quantum gauge theory. Of course the instanton physics in pure non-abelian gauge theories such as Yang-Mills (YM) theory formulated in flat Euclidean space has been studied thoroughly thus far. Its study in non-trivial but physically meaningful gravitational fields, however, has been extremely incomplete. Indeed, the strength of gravity well below the Planck scale is negligibly small compared to those of elementary particle interactions described by non-abelian gauge theories. Thus one might overlook the effects of gravity on non-perturbative regime of non-abelian gauge theories such as the physics of instanton. Nevertheless, no matter how weak the relative strength of the background gravity is, as long as the gravity carries non-trivial topology, it may have profound effects on the structure of gauge theory instantons since these instantons are topological objects linked to the topology-changing processes. Therefore in the present work, we would like to explore how the topological properties of the YM theory (or more precisely, of the YM instanton solution) are dictated by the non-trivial topology of the gravitational field with which it interacts. Being an issue of great physical interest and importance, quite a few serious study along this line have appeared in the literature but they were restricted to the background gravitational field with high degree of isometry such as the Euclideanized Schwarzschild geometry [2] or the Euclidean de Sitter space [3]. Even the works involving more general background spacetimes including gravitational instantons

(GI) were mainly confined to the case of asymptotically- locally-Euclidean (ALE) spaces which is one particular such GI and employed rather indirect and mathmatically-oriented solution generating methods such as the ADHM construction [14]. Recently, we [4] have proposed a "simply physical" and hence perhaps the most direct algorithm for generating the YM instanton solutions in all species of known GI. Particularly, in [4] this new algorithm has been applied to the construction of solutions to (anti) self-dual YM equation in the background of Taub-NUT and Eguchi-Hanson metrics which are the best-known such GI. In the present work, we would like to complete our discussion on this issue by providing a detailed presentation of our algorithm and applying it to practically all the GI known. The careful physicai interpretation of the solutions obtained eventually to determine their nature will also be given in this work. The essence of this method lies in writing the (anti) self-dual YM equation by employing truly relevant ansắtz for the YM gauge connection and then directly solving it. To demonstrate how simple in method and powerful in applicability it is, we then apply this algorithm to the case of (anti) self-dual YM equations in almost all of known GI and find the YM instanton solutions in their backgrounds. In particular, the actual YM instanton solution in the background of Taub-NUT (which is asymptotically-locally-flat (ALF) rather than ALE), Fubini-Study (on CP^2), and de Sitter (on S^4) metrics are constructed for the first time in this work. Interestingly, the solutions to (anti) self-dual YM equation turn out to be the rather exotic type of instanton configurations which are everywhere non-singular having *finite* YM action but sharing some features with meron solutions [11] such as their typical structure and generally *fractional* topological charge values carried by them. Namely, the YM instanton solution that we shall discuss in the background of GI in this work exhibit characteristics which are mixture of those of typical instanton and typical meron. Thus at this point, it seems relevant to briefly review the essential nature of meron solution. For detailed description of meron, we refer the reader to some earlier works [10,11]. First, recall that the standard BPST [1] SU(2) YM instanton solution in flat space takes the form $A^a_\mu = 2\eta^a_{\mu\nu}[x^\nu/(r^2+\lambda^2)]$ with $\eta^a_{\mu\nu}$ and λ being the 'tHooft tensor [5] and the size of the instanton respectively while the meron solution which is another non-trivial solution to the second order YM field equation found long ago by De Alfaro, Fubini, and Furlan [10] takes the form $A^a_\mu = \eta^a_{\mu\nu}(x^\nu/r^2)$. Since the pure (vacuum) gauge having vanishing field strength is given by $A^a_\mu = 2\eta^a_{\mu\nu}(x^\nu/r^2)$, the standard instanton solution interpolates between the trivial vacuum $A^a_\mu = 0$ at $r = 0$ and another vacuum represented by this pure gauge above at $r \to \infty$ and the meron solution can be thought of as a "half a vacuum gauge". Unlike the instanton solution, however, the meron solution only solves the second order YM field equation and fails to solve the first order (anti) self-dual equation. As is apparent from their structures given above, the meron is an unstable solution in that it is singular at its center $r = 0$ and at $r = \infty$ while the ordinary instanton solution exhibits no singular behavior. As was pointed out originally by De Alfaro et al. [10], in contrast to instantons whose topological charge density is a smooth function of x, the topological charge density of merons vanishes everywhere except at its center, i.e., the singular point, such that its volume integral is half unit of topological charge 1/2. And curiously enough, half-integer topological charge seems to be closely related to the confinement in the Schwinger model [11]. It is also amusing to note that a "time slice" through the origin, i.e., $x_4 = 0$ of the meron configuration yields a $SU(2)$ Wu-Yang monopole [11]. Lastly, the Euclidean meron action diverges logarithmically and perhaps needs some regularization whereas the standard YM instanton has finite action.

We now recall some generic features of gravitational instantons. In the loose sense, GI may be defined as a positive-definite metrics $g_{\mu\nu}$ on a complete and non-singular manifold satisfying the Euclidean Einstein equations and hence constituting the stationary points of the gravity action in Euclidean path integral for quantum gravity. But in the stricter sense [5,6], they are the metric solutions to the Euclidean Einstein equations having (anti) self-dual Riemann tensor

$$\tilde{R}_{abcd} = \frac{1}{2}\epsilon_{ab}{}^{ef}R_{efcd} = \pm R_{abcd} \tag{1}$$

(say, with indices written in non-coordinate orthonormal basis) and include only two families of solutions in a rigorous sense ; the Taub-NUT metric [7] and the Eguchi-Hanson instanton

[8]. In the loose sense, however, there are several solutions to Euclidean Einstein equations that can fall into the category of GI.

II. New algorithm for solutions to (anti) self-dual YM equation

We now begin with the action governing our system, i.e., the Einstein-Yang-Mills (EYM) theory given by

$$I_{EYM} = \int_M d^4x\sqrt{g}\left[\frac{-1}{16\pi}R + \frac{1}{4g_c^2}F^a_{\mu\nu}F^{a\mu\nu}\right] - \int_{\partial M} d^3x\sqrt{h}\frac{1}{8\pi}K \tag{2}$$

where $F^a_{\mu\nu}$ is the field strength of the YM gauge field A^a_μ with $a = 1, 2, 3$ being the SU(2) group index and g_c being the gauge coupling constant. The Gibbons-Hawking term on the boundary ∂M of the manifold M is also added and h is the metric induced on ∂M and K is the trace of the second fundamental form on ∂M. Then by extremizing this action with respect to the metric $g_{\mu\nu}$ and the YM gauge field A^a_μ, one gets the following classical field equations respectively

$$\begin{aligned}
&R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}, \\
&T_{\mu\nu} = \frac{1}{g_c^2}\left[F^a_{\mu\alpha}F^{a\alpha}_{\nu} - \frac{1}{4}g_{\mu\nu}(F^a_{\alpha\beta}F^{a\alpha\beta})\right], \\
&D_\mu\left[\sqrt{g}F^{a\mu\nu}\right] = 0, \quad D_\mu\left[\sqrt{g}\tilde{F}^{a\mu\nu}\right] = 0
\end{aligned} \tag{3}$$

where we added Bianchi identity in the last line and $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc}A^b_\mu A^c_\nu$, $D^{ac}_\mu = \partial_\mu\delta^{ac} + \epsilon^{abc}A^b_\mu$ and $A_\mu = A^a_\mu(-iT^a)$, $F_{\mu\nu} = F^a_{\mu\nu}(-iT^a)$ with $T^a = \tau^a/2$ $(a = 1, 2, 3)$ being the SU(2) generators and finally $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}F_{\alpha\beta}$ is the (Hodge) dual of the field strength tensor. We now seek solutions $(g_{\mu\nu},\ A^a_\mu)$ of the coupled EYM equations given above in Euclidean signature obeying the (anti) self-dual equation in the YM sector

$$F^{\mu\nu} = g^{\mu\lambda}g^{\nu\sigma}F_{\lambda\sigma} = \pm\frac{1}{2}\epsilon_c^{\mu\nu\alpha\beta}F_{\alpha\beta} \tag{4}$$

where $\epsilon_c^{\mu\nu\alpha\beta} = \epsilon^{\mu\nu\alpha\beta}/\sqrt{g}$ is the curved spacetime version of totally antisymmetric tensor. As was noted in [2,3], in Euclidean signature, the YM energy-momentum tensor vanishes identically for YM fields satisfying this (anti) self-duality condition. This point is of central importance and can be illustrated briefly as follows. Under the Hodge dual transformation, $F^a_{\mu\nu} \to \tilde{F}^a_{\mu\nu}$, the YM energy-momentum tensor $T_{\mu\nu}$ given in eq.(3) above is invariant normally in Lorentzian signature. In Euclidean signature, however, its sign flips, i.e., $\tilde{T}_{\mu\nu} = -T_{\mu\nu}$. As a result, for YM fields satisfying the (anti) self-dual equation in Euclidean signature such as the instanton solution, $F^a_{\mu\nu} = \pm\tilde{F}^a_{\mu\nu}$, it follows that $T_{\mu\nu} = -\tilde{T}_{\mu\nu} = -T_{\mu\nu}$, namely the YM energy-momentum tensor vanishes identically, $T_{\mu\nu} = 0$. This, then, indicates that the YM field now does not disturb the geometry while the geometry still does have effects on the YM field. Consequently the geometry, which is left intact by the YM field, effectively serves as a "background" spacetime which can be chosen somewhat at our will (as long as it satisfies the vacuum Einstein equation $R_{\mu\nu} = 0$) and here in this work, we take it to be the gravitational instanton. Loosely speaking, all the typical GI, including Taub-NUT metric and Eguchi-Hanson solution, possess the same topology $R \times S^3$ and similar metric structures. Of course in a stricter sense, their exact topologies can be distinguished, say, by different Euler numbers and Hirzebruch signatures [5,6]. Particularly, in terms of the concise basis 1-forms, the metrics of these GI can be written as [5,6]

$$\begin{aligned} ds^2 &= c_r^2 dr^2 + c_1^2\left(\sigma_1^2 + \sigma_2^2\right) + c_3^2\sigma_3^2 \\ &= c_r^2 dr^2 + \sum_{a=1}^{3} c_a^2\left(\sigma^a\right)^2 = e^A \otimes e^A \end{aligned} \tag{5}$$

where $c_r = c_r(r)$, $c_a = c_a(r)$, $c_1 = c_2 \neq c_3$ and the orthonormal basis 1-form e^A is given by

$$e^A = \left\{e^0 = c_r dr, \quad e^a = c_a \sigma^a\right\} \tag{6}$$

and $\{\sigma^a\}$ $(a = 1, 2, 3)$ are the left-invariant 1-forms satisfying the SU(2) Maurer-Cartan structure equation

$$d\sigma^a = -\frac{1}{2}\epsilon^{abc}\sigma^b \wedge \sigma^c. \tag{7}$$

They form a basis on the S^3 section of the geometry and hence can be represented in terms of 3-Euler angles $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$, and $0 \le \psi \le 4\pi$ parametrizing S^3 as

$$
\begin{aligned}
\sigma^1 &= -\sin\psi d\theta + \cos\psi \sin\theta d\phi, \\
\sigma^2 &= \cos\psi d\theta + \sin\psi \sin\theta d\phi, \qquad (8) \\
\sigma^3 &= -d\psi - \cos\theta d\phi.
\end{aligned}
$$

Now in order to construct exact YM instanton solutions in the background of these GI, we now choose the relevant ansätz for the YM gauge potential and the SU(2) gauge fixing. And in doing so, our general guideline is that the YM gauge field ansätz should be endowed with the symmetry inherited from that of the background geometry, the GI. Thus we first ask what kind of isometry these GI possess. As noted above, all the typical GI possess the topology of $R \times S^3$. The geometrical structure of the S^3 section, however, is not that of perfectly "round" S^3 but rather, that of "squashed" S^3. In order to get a closer picture of this squashed S^3, we notice that the r =constant slices of these GI can be viewed as U(1) fibre bundles over $S^2 \sim CP^1$ with the line element

$$
d\Omega_3^2 = c_1^2\left(\sigma_1^2 + \sigma_2^2\right) + c_3^2\sigma_3^2 = c_1^2 d\Omega_2^2 + c_3^2\left(d\psi + B\right)^2 \qquad (9)
$$

where $d\Omega_2^2 = (d\theta^2 + \sin^2\theta d\phi^2)$ is the metric on unit S^2, the base manifold whose volume form Ω_2 is given by $\Omega_2 = dB$ as $B = \cos\theta d\phi$ and ψ then is the coordinate on the U(1)$\sim S^1$ fibre manifold. Now then the fact that $c_1 = c_2 \neq c_3$ indicates that the geometry of this fibre bundle manifold is not that of round S^3 but that of squashed S^3 with the squashing factor given by (c_3/c_1). And further, it is squashed along the U(1) fibre direction. Thus this failure for the geometry to be that of exactly round S^3 keeps us from writing down the associated ansätz for the YM gauge potential right away. Apparently, if the geometry were that of round S^3, one would write down the YM gauge field ansätz as $A^a = f(r)\sigma^a$ [3] with $\{\sigma^a\}$ being the left-invariant 1-forms introduced earlier. The rationale for this choice can be stated briefly as follows. First, since the r =constant sections of the background space have the geometry of round S^3 and hence possess the SO(4)-isometry, one would look for the SO(4)-invariant YM gauge connection ansätz as well. Next, noticing that both the r =constant sections of the frame manifold and the SU(2) YM group manifold possess the geometry of round S^3, one may naturally choose the left-invariant 1-forms $\{\sigma^a\}$ as the "common" basis for both manifolds. Thus this YM gauge connection ansätz, $A^a = f(r)\sigma^a$ can be thought of as a hedgehog-type ansätz where the group-frame index mixing is realized in a simple manner [3]. Then coming back to our present interest, namely the GI given in eq.(5), in r =constant sections, the SO(4)-isometry is partially broken down to that of SO(3) by the squashedness along the U(1) fibre direction to a degree set by the squashing factor (c_3/c_1). Thus now our task became clearer and it is how to encode into the YM gauge connection ansätz this particular type of SO(4)-isometry breaking coming from the squashed S^3. Interestingly, a clue to this puzzle can be drawn from the work of Eguchi and Hanson [9] in which they constructed abelian instanton solution in Euclidean Taub-NUT metric (namely the abelian gauge field with (anti)self-dual field strength with respect to this metric). To get right to the point, the working ansätz they employed for the abelian gauge field to yield (anti)self-dual field strength is to align the abelian gauge connection 1-form along the squashed direction, i.e., along the U(1) fibre direction, $A = g(r)\sigma^3$. This choice looks quite natural indeed. After all, realizing that embedding of a gauge field in a geometry with high degree of isometry is itself an isometry (more precisely isotropy)-breaking action, it would be natural to put it along the direction in which part of the isometry is already broken. Finally therefore, putting these two pieces of observations carefully together, now we are in the position to suggest the relevant ansätz for the YM gauge connection 1-form in these GI and it is

$$A^a = f(r)\sigma^a + g(r)\delta^{a3}\sigma^3 \tag{10}$$

which obviously would need no more explanatory comments except that in this choice of the ansätz, it is implicitly understood that the gauge fixing $A_r = 0$ is taken. From this point on, the construction of the YM instanton solutions by solving the (anti)self-dual equation given in eq.(4) is straightforward. To sketch briefly the computational algorithm, first we obtain the YM field strength 2-form (in orthonormal basis) via exterior calculus (since the YM gauge connection ansätz is given in left-invariant 1-forms) as $F^a = (F^1, F^2, F^3)$ where

$$
\begin{aligned}
F^1 &= \frac{f'}{c_r c_1}(e^0 \wedge e^1) + \frac{f[(f-1)+g]}{c_2 c_3}(e^2 \wedge e^3), \\
F^2 &= \frac{f'}{c_r c_2}(e^0 \wedge e^2) + \frac{f[(f-1)+g]}{c_3 c_1}(e^3 \wedge e^1), \\
F^3 &= \frac{(f'+g')}{c_r c_3}(e^0 \wedge e^3) + \frac{[f(f-1)-g]}{c_1 c_2}(e^1 \wedge e^2)
\end{aligned} \tag{11}
$$

from which we can read off the (anti)self-dual equation to be

$$
\pm \frac{f'}{c_r c_1} = \frac{f[(f-1)+g]}{c_2 c_3}, \qquad \pm \frac{(f'+g')}{c_r c_3} = \frac{[f(f-1)-g]}{c_1 c_2} \tag{12}
$$

where the prime denotes the derivative with respect to r. After some manipulation, these (anti) self-dual equation can be cast to a more practical form

$$
(\ln f)'' + \left[\left(\frac{c_3}{c_r}\right)'\left(\frac{c_r}{c_3}\right) \pm \left(\frac{c_r c_3}{c_1 c_2}\right)\right](\ln f)' = \left(\frac{c_r}{c_1}\right)^2 [f^2 - 1], \tag{13}
$$

$$
h = \left[1 \pm \left(\frac{c_3}{c_r}\right)(\ln f)'\right] \tag{14}
$$

where $h(r) = f(r) + g(r)$ and "+" for self-dual and "$-$" for anti-self-dual equation and we have only a set of two equations as $c_1 = c_2$. Now the remaining computational algorithm is, for each GI corresponding to particular choice of $e^A = \{e^0 = c_r dr, \quad e^a = c_a \sigma^a\}$, first eqs.(13) and (14), if admit solutions, give $f(r)$ and $g(r)$ respectively and from which, next the YM instanton solutions in eq.(10) and their (anti) self-dual field strength in eq.(11) can be obtained. At this point, it is interesting to realize that actually there are other avenues to constructing the YM instanton solutions of different species from that given in eq.(10) in these GI. To state once again, in $r = constant$ sections of GI, since the SO(4)-isometry is partially broken by the squashedness of S^3 along the U(1) fibre direction set by σ^3 in eq.(9), this particular direction set by σ_3 can be thought of as a kind of that of *principal axis*. Note also that exactly to the same degree this U(1) fibre direction set by σ^3 stands out, the other two directions set by σ^1 and σ^2 respectively, may be regarded as being special. Thus one might as well want to align the YM gauge connection solely along the direction set by σ^3 or along the direction set by σ^1 or σ^2. And this can only be done when one abandons the non-abelian structure in the YM gauge field and writes its ansätz in the form

$$A^a = g(r)\delta^{a3}\sigma^3 \quad \text{or} \quad A^a = g(r)\delta^{a1(2)}\sigma^{1(2)}$$

respectively. Then YM instanton solutions of these species should essentially be equivalent to the abelian instantons of the Eguchi-Hanson-type mentioned earlier and as such they, if exist, should clearly be totally different kinds of instanton solutions that cannot be related to the standard YM instantons given in eq.(10) via any gauge transformation whatsoever. For this reason, we shall call them "abelianized" YM instanton solutions and attempt to construct them in this work as well. The field strength and the (anti) self-dual equations associated with these abelianized YM instantons are then given respectively by

$$\begin{aligned} F^a &= \left[\frac{g'}{c_r c_3}(e^0 \wedge e^3) - \frac{g}{c_1 c_2}(e^1 \wedge e^2)\right]\delta^{a3}, \\ \pm (\ln g)' &= -\left(\frac{c_r c_3}{c_1 c_2}\right) \quad \text{for} \quad A^a = g(r)\delta^{a3}\sigma^3 \end{aligned} \tag{15}$$

and

$$\begin{aligned} F^a &= \left[\frac{g'}{c_r c_1}(e^0 \wedge e^1) - \frac{g}{c_2 c_3}(e^2 \wedge e^3)\right]\delta^{a1}, \\ \pm (\ln g)' &= -\left(\frac{c_r}{c_3}\right) \quad \text{for} \quad A^a = g(r)\delta^{a1}\sigma^1 \end{aligned} \tag{16}$$

and similarly for $A^a = g(r)\delta^{a2}\sigma^2$. And in the above equations, "+" for self-dual and "−" for anti-self-dual equation. We now present both the "standard" and "abelianized" YM instanton solutions of the forms given in eqs.(10),(15), and (16) for each of the GI.

III. Application of the algorithm to various GI backgrounds

In this section, in order to exhibit how simple in method and how powerful in applicability this new algorithm of ours really is, we shall apply the algorithm to the cases of Taub-NUT (TN), Eguchi-Hanson (EH), Fubini-Study (FS), Taub-bolt (TB), and de Sitter GI backgrounds and find the solutions to (anti) self-dual YM equations in these GI.

(1) YM instanton in Taub-NUT (TN) metric background

The TN GI solution written in the metric form given in eq.(5) amounts to

$$c_r = \frac{1}{2}\left[\frac{r+m}{r-m}\right]^{1/2}, \quad c_1 = c_2 = \frac{1}{2}\left[r^2 - m^2\right]^{1/2}, \quad c_3 = m\left[\frac{r-m}{r+m}\right]^{1/2}$$

and it is a solution to Euclidean vacuum Einstein equation $R_{\mu\nu} = 0$ for $r \geq m$ with self-dual Riemann tensor. The apparent singularity at $r = m$ can be removed by a coordinate redefinition and is a 'nut' (in terminology of Gibbons and Hawking [6]) at which the isometry generated by the Killing vector $(\partial/\partial\psi)$ has a zero-dimensional fixed point set. The boundary of TN metric at $r \to \infty$ is S^3. And this TN instanton is an asymptotically-locally-flat (ALF) metric.

(i) Standard YM instanton solution

It turns out that only the anti-self-dual equation $F^a = -\tilde{F}^a$ admits a non-trivial solution and it is $A^a = (A^1, A^2, A^3)$ where

$$A^1 = \pm 2\frac{(r-m)^{1/2}}{(r+m)^{3/2}}e^1, \quad A^2 = \pm 2\frac{(r-m)^{1/2}}{(r+m)^{3/2}}e^2, \quad A^3 = \frac{(r+3m)}{m}\frac{(r-m)^{1/2}}{(r+m)^{3/2}}e^3 \tag{17}$$

and $F^a = (F^1, F^2, F^3)$ where

$$\begin{aligned} F^1 &= \pm\frac{8m}{(r+m)^3}\left(e^0 \wedge e^1 - e^2 \wedge e^3\right), \quad F^2 = \pm\frac{8m}{(r+m)^3}\left(e^0 \wedge e^2 - e^3 \wedge e^1\right), \\ F^3 &= \frac{16m}{(r+m)^3}\left(e^0 \wedge e^3 - e^1 \wedge e^2\right). \end{aligned} \tag{18}$$

It is interesting to note that this YM field strength and the Ricci tensor of the background TN GI are proportional as $|F^a| = 2|R^0_a|$ except for opposite self-duality, i.e.,

$$\begin{aligned} R^0_1 &= -R^2_3 = \frac{4m}{(r+m)^3}\left(e^0 \wedge e^1 + e^2 \wedge e^3\right), \quad R^0_2 = -R^3_1 = \frac{4m}{(r+m)^3}\left(e^0 \wedge e^2 + e^3 \wedge e^1\right), \\ R^0_3 &= -R^1_2 = -\frac{8m}{(r+m)^3}\left(e^0 \wedge e^3 + e^1 \wedge e^2\right). \end{aligned} \tag{19}$$

(ii) Abelianized YM instanton along the direction set by σ^3

Both the self-dual and anti-self-dual equations admit non-trivial solutions and they are, in orthonormal basis,

$$\begin{aligned} A^a &= k\left(\frac{r+m}{r-m}\right)\delta^{a3}\sigma^3 = \frac{k}{m}\left(\frac{r+m}{r-m}\right)^{3/2}\delta^{a3}e^3, \\ F^a &= -\frac{4k}{(r-m)^2}\left[(e^0 \wedge e^3) + (e^1 \wedge e^2)\right]\delta^{a3} \end{aligned} \tag{20}$$

for the solution to self-dual equation and

$$A^a = k\left(\frac{r-m}{r+m}\right)\delta^{a3}\sigma^3 = \frac{k}{m}\left(\frac{r-m}{r+m}\right)^{1/2}\delta^{a3}e^3, \tag{21}$$
$$F^a = \frac{4k}{(r+m)^2}\left[(e^0\wedge e^3)-(e^1\wedge e^2)\right]\delta^{a3}$$

for the solution to anti-self-dual equation. In these solutions, k is an arbitrary constant.

(iii) Abelianized YM instanton along the direction set by σ^1

Again, both the self-dual and anti-self-dual equations admit non-trivial solutions and they are

$$A^a = \frac{k}{(r-m)}e^{-r/2m}\delta^{a1}\sigma^1 = \frac{2k}{(r+m)^{1/2}(r-m)^{3/2}}e^{-r/2m}\delta^{a1}e^1, \tag{22}$$
$$F^a = -\frac{2k}{m}\frac{e^{-r/2m}}{(r-m)^2}\left[(e^0\wedge e^1)+(e^2\wedge e^3)\right]\delta^{a1}$$

for the solution to self-dual equation and

$$A^a = k(r-m)e^{r/2m}\delta^{a1}\sigma^1 = 2k\left(\frac{r-m}{r+m}\right)^{1/2}e^{r/2m}\delta^{a1}e^1, \tag{23}$$
$$F^a = \frac{2k}{m}e^{r/2m}\left[(e^0\wedge e^1)-(e^2\wedge e^3)\right]\delta^{a1}$$

for the solution to anti-self-dual equation. This solution, however, is *not* physical and hence should be dropped as it blows up as $r\to\infty$.

(2) YM instanton in Eguchi-Hanson (EH) metric background

The EH GI solution amounts to

$$c_r = \left[1-\left(\frac{a}{r}\right)^4\right]^{-1/2}, \quad c_1 = c_2 = \frac{1}{2}r, \quad c_3 = \frac{1}{2}r\left[1-\left(\frac{a}{r}\right)^4\right]^{1/2}$$

and again it is a solution to Euclidean vacuum Einstein equation $R_{\mu\nu}=0$ for $r\geq a$ with self-dual Riemann tensor. $r=a$ is just a coordinate singularity that can be removed by a coordinate redefinition provided that now ψ is identified with period 2π rather than 4π and is a 'bolt' (in terminology of Gibbons and Hawking [6]) where the action of the Killing field $(\partial/\partial\psi)$ has a two-dimensional fixed point set. Note that for an ordinary S^3, the range for the Euler angle ψ would be $0\leq\psi\leq 4\pi$. Thus demanding $0\leq\psi\leq 2\pi$ instead to remove the bolt singularity at $r = a$ amounts to identifying points antipodal with respect to the origin and this, in turn, implies that the boundary of EH at $r \to \infty$ is the real projective space $RP^3 = S^3/Z_2$. Besides, this EH instanton is an asymptotically-locally-Euclidean (ALE) metric.

(i) Standard YM instanton solution

In this time, only the self-dual equation $F^a = +\tilde{F}^a$ admits a non-trivial solution and it is $A^a = (A^1, A^2, A^3)$ where

$$A^1 = \pm\frac{2}{r}\left[1-\left(\frac{a}{r}\right)^4\right]^{1/2} e^1, \quad A^2 = \pm\frac{2}{r}\left[1-\left(\frac{a}{r}\right)^4\right]^{1/2} e^2, \quad A^3 = \frac{2}{r}\frac{\left[1+\left(\frac{a}{r}\right)^4\right]}{\left[1-\left(\frac{a}{r}\right)^4\right]^{1/2}} e^3 \qquad (24)$$

and $F^a = (F^1, F^2, F^3)$ where

$$\begin{aligned} F^1 &= \pm\frac{4}{r^2}\left(\frac{a}{r}\right)^4\left(e^0\wedge e^1 + e^2\wedge e^3\right), \quad F^2 = \pm\frac{4}{r^2}\left(\frac{a}{r}\right)^4\left(e^0\wedge e^2 + e^3\wedge e^1\right), \\ F^3 &= -\frac{8}{r^2}\left(\frac{a}{r}\right)^4\left(e^0\wedge e^3 + e^1\wedge e^2\right). \end{aligned} \qquad (25)$$

Again it is interesting to realize that this YM field strength and the Ricci tensor of the background EH GI are proportional as $|F^a| = 2|R^0_a|$, i.e.,

$$\begin{aligned} R^0_1 &= -R^2_3 = \frac{2}{r^2}\left(\frac{a}{r}\right)^4\left(-e^0\wedge e^1 + e^2\wedge e^3\right), \quad R^0_2 = -R^3_1 = \frac{2}{r^2}\left(\frac{a}{r}\right)^4\left(-e^0\wedge e^2 + e^3\wedge e^1\right), \\ R^0_3 &= -R^1_2 = -\frac{4}{r^2}\left(\frac{a}{r}\right)^4\left(-e^0\wedge e^3 + e^1\wedge e^2\right). \end{aligned} \qquad (26)$$

It is also interesting to note that this YM instanton solution particularly in EH background (which is ALE) obtained by directly solving the self-dual equation can also be "constructed" by simply identifying $A^a = \pm 2\omega^0_a$ (where $\omega^0_a = (\epsilon_{abc}/2)\omega^{bc}$ are the spin connection of EH metric) and hence $F^a = \pm 2R^0_a$ as was noticed by [13] but in the string theory context with different motivation. This construction of solution via a simple identification of gauge field connection with the spin connection, however, works only in ALE backgrounds such as EH metric and generally fails as is manifest in the previous TN background case (which is ALF, not ALE) in which $A^a \neq \pm 2\omega^0_a$ but still $F^a = \pm 2R^0_a$. Thus the method presented here by first writing (by employing a relevant ansätz for the YM gauge connection given in eq.(10))

and directly solving the (anti) self-dual equation looks to be the algorithm for generating the solution with general applicability to all species of GI in a secure and straightforward manner. In this regard, the method for generating YM instanton solutions to (anti) self-dual equation in all known GI backgrounds proposed here in this work can be contrasted to earlier works in the literature [15] discussing the construction of YM instantons mainly in the background of ALE GI via indirect methods such as that of ADHM [14].

(ii) Abelianized YM instanton along the direction set by σ^3

Both the self-dual and anti-self-dual equations admit non-trivial solutions and they are

$$\begin{aligned} A^a &= \frac{k}{r^2}\delta^{a3}\sigma^3 = \frac{2k}{r^3}\left[1-\left(\frac{a}{r}\right)^4\right]^{-1/2}\delta^{a3}e^3, \\ F^a &= -\frac{4k}{r^4}\left[(e^0\wedge e^3)+(e^1\wedge e^2)\right]\delta^{a3} \end{aligned} \tag{27}$$

for the solution to self-dual equation and

$$\begin{aligned} A^a &= kr^2\delta^{a3}\sigma^3 = 2kr\left[1-\left(\frac{a}{r}\right)^4\right]^{-1/2}\delta^{a3}e^3, \\ F^a &= 4k\left[(e^0\wedge e^3)-(e^1\wedge e^2)\right]\delta^{a3} \end{aligned} \tag{28}$$

for the solution to anti-self-dual equation. In these solutions, k is an arbitrary constant. Note, however, that this solution to the anti-self-dual equation is *unphysical* and thus should be dropped as it fails to represent a localized soliton configuration.

(iii) Abelianized YM instanton along the direction set by σ^1

Again, both the self-dual and anti-self-dual equations admit non-trivial solutions and they are

$$\begin{aligned} A^a &= \frac{k}{\sqrt{r^4-a^4}}\delta^{a1}\sigma^1 = \frac{2k}{r\sqrt{r^4-a^4}}\delta^{a1}e^1, \\ F^a &= -\frac{4k}{(r^4-a^4)}\left[(e^0\wedge e^1)+(e^2\wedge e^3)\right]\delta^{a1} \end{aligned} \tag{29}$$

for the solution to self-dual equation and

$$\begin{aligned} A^a &= k\sqrt{r^4-a^4}\delta^{a1}\sigma^1 = \frac{2k}{r}\sqrt{r^4-a^4}\delta^{a1}e^1, \\ F^a &= 4k\left[(e^0\wedge e^1)-(e^2\wedge e^3)\right]\delta^{a1} \end{aligned} \tag{30}$$

for the solution to anti-self-dual equation. Again, this solution is *not* physical and hence should be discarded as it fails to represent a localized soliton configuration.

(3) YM instanton in Fubini-Study (FS) metric on CP^2 background

Lastly, the FS (on complex projective plane CP^2) gravitational instanton solution corresponds to

$$c_r = \left[1+\frac{1}{6}\Lambda r^2\right]^{-1}, \quad c_1 = c_2 = \frac{r}{2}\left[1+\frac{1}{6}\Lambda r^2\right]^{-1/2}, \quad c_3 = \frac{r}{2}\left[1+\frac{1}{6}\Lambda r^2\right]^{-1}$$

where Λ is the (positive) cosmological constant and it is a solution to the Euclidean Einstein equation $R_{\mu\nu} = 8\pi\Lambda g_{\mu\nu}$. As such, this FS metric is a "compact" gravitational instanton (i.e., instanton of finite volume) with no boundary and is everywhere regular up to the fact that a close inspection [5,6] reveals that at $r = 0$, there is a removable nut singularity while at $r \to \infty$, we have a bolt singularity which is removable provided $0 \le \psi \le 4\pi$. Besides, unlike the previous TN and EH instantons which have self-dual Riemann tensors $R_{\mu\nu\alpha\beta} = \tilde{R}_{\mu\nu\alpha\beta}$, this FS instanton possesses self-dual Weyl tensor $C_{\mu\nu\alpha\beta} = \tilde{C}_{\mu\nu\alpha\beta}$.

(i) Standard YM instanton solution

Only the self-dual equation $F^a = +\tilde{F}^a$ admits a non-trivial solution and the corresponding solution and the associated self-dual field strength are given by

$$A^1 = \pm\frac{2}{r}e^1, \quad A^2 = \pm\frac{2}{r}e^2, \quad A^3 = \frac{2}{r}(1+\frac{1}{12}\Lambda r^2)e^3 \tag{31}$$

and $F^a = (F^1, F^2, F^3)$ where

$$\begin{aligned} F^1 &= \pm\frac{\Lambda}{3}\left(e^0\wedge e^1 + e^2\wedge e^3\right), \quad F^2 = \pm\frac{\Lambda}{3}\left(e^0\wedge e^2 + e^3\wedge e^1\right), \\ F^3 &= -\frac{\Lambda}{3}\left(e^0\wedge e^3 + e^1\wedge e^2\right). \end{aligned} \tag{32}$$

Again it is interesting to contrast this YM field strength with the Ricci tensor of the background FS GI given by

$$\begin{aligned} R^0_1 &= -R^2_3 = \frac{\Lambda}{6}\left(e^0\wedge e^1 - e^2\wedge e^3\right), \quad R^0_2 = -R^3_1 = \frac{\Lambda}{6}\left(e^0\wedge e^2 - e^3\wedge e^1\right), \\ R^0_3 &= \frac{\Lambda}{3}\left(2e^0\wedge e^3 + e^1\wedge e^2\right), \quad R^1_2 = \frac{\Lambda}{3}\left(e^0\wedge e^3 + 2e^1\wedge e^2\right) \end{aligned} \tag{33}$$

which, unlike the TN and EH cases, *fails* to obey the relation $|F^a| = 2|R^0_a|$ presumably because the FS solution fails to have self-dual Riemann tensor. Here it seems worthy of note that since the background FS metric is a compact instanton and hence has a finite volume, one needs not worry about the possible divergence of the field energy upon integration over the volume. Namely, this instanton solution is a legitimate, physical solution.

(ii) Abelianized YM instanton along the direction set by σ^3

Again, both the self-dual and anti-self-dual equations admit non-trivial solutions and they are

$$\begin{aligned} A^a &= \frac{6k}{\Lambda r^2}(1+\frac{1}{6}\Lambda r^2)\delta^{a3}\sigma^3 = \frac{12k}{\Lambda r^3}(1+\frac{1}{6}\Lambda r^2)^2\delta^{a3}e^3, \\ F^a &= -\frac{24k}{\Lambda r^4}(1+\frac{1}{6}\Lambda r^2)^2\left[(e^0\wedge e^3)+(e^1\wedge e^2)\right]\delta^{a3} \end{aligned} \tag{34}$$

for the solution to self-dual equation and

$$\begin{aligned} A^a &= \frac{k\Lambda}{6}r^2(1+\frac{1}{6}\Lambda r^2)^{-1}\delta^{a3}\sigma^3 = \frac{k}{3}\Lambda r\delta^{a3}e^3, \\ F^a &= \frac{2k\Lambda}{3}\left[(e^0\wedge e^3)-(e^1\wedge e^2)\right]\delta^{a3} \end{aligned} \tag{35}$$

for the solution to anti-self-dual equation. In these solutions, k is again an arbitrary constant.

(iii) Abelianized YM instanton along the direction set by σ^1

$$\begin{aligned} A^a &= \frac{k}{r^2}\delta^{a1}\sigma^1 = \frac{2k}{r^3}(1+\frac{1}{6}\Lambda r^2)^{1/2}\delta^{a1}e^1, \\ F^a &= -\frac{4k}{r^4}(1+\frac{1}{6}\Lambda r^2)^{3/2}\left[(e^0\wedge e^1)+(e^2\wedge e^3)\right]\delta^{a1} \end{aligned} \tag{36}$$

for the solution to self-dual equation and

$$\begin{aligned} A^a &= kr^2\delta^{a1}\sigma^1 = 2kr(1+\frac{1}{6}\Lambda r^2)^{1/2}\delta^{a1}e^1, \\ F^a &= 4k(1+\frac{1}{6}\Lambda r^2)^{3/2}\left[(e^0\wedge e^1)-(e^2\wedge e^3)\right]\delta^{a1}. \end{aligned} \tag{37}$$

for the solution to anti-self-dual equation.

And this completes the presentation of all non-trivial YM instanton solutions in three families of gravitational instantons. We discussed earlier in the introduction the classification of gravitational instantons [5,6]. And the three families of gravitational instantons, TN, EH, and FS metrics fall into the class of instanton solutions in the stricter sense as they have (anti) self-dual Riemann or Weyl tensor. In this classification, all the other gravitational instantons discovered thus far can be thought of as being instanton solutions in the loose sense as they all fail to satisfy (anti) self-dual condition for Riemann or Weyl tensor although still are the solutions to the Euclidean Einstein equation with or without the cosmological constant. Therefore for the sake of completeness of our study, here we also provide explicit YM instanton solutions in the background of other species of gravitational instantons in the loose sense. And particularly, we consider the Taub-bolt metric [10] and the de Sitter metric on S^4 [5,6].

(4) YM instanton in Taub-bolt (TB) metric background

This TB GI solutin written in the metric form given in eq.(5) corresponds to

$$c_r = \left[\frac{2(r^2 - N^2)}{2r^2 - 5Nr + 2N^2}\right]^{1/2}, \quad c_1 = c_2 = \left[r^2 - N^2\right]^{1/2}, \quad c_3 = 2N\left[\frac{2r^2 - 5Nr + 2N^2}{2(r^2 - N^2)}\right]^{1/2}$$

and it is a solution to Euclidean vacuum Einstein equation $R_{\mu\nu} = 0$ for $r \geq 2N$. Again, in terminology of Gibbons and Hawking [5,6], $r = 2N$ is a 'bolt' singularity that can be removed by a coordinate redefinition. As stated, although neither its Riemann nor Weyl tensor is (anti) self-dual, it is, like the TN-metric, another asymptotically locally-flat (ALF) instanton.

(i) Standard YM instanton solution

Unlike the ones belonging to the class of instanton solutions in the stricter sense, i.e., TN, EH, and FS metrics, neither self-dual nor anti-self-dual equation $F^a = \pm\tilde{F}^a$ in this TB-metric background admits any non-trivial solution.

(ii) Abelianized YM instanton along the direction set by σ^3

Both the self-dual and anti-self-dual equations admit non-trivial solutions and they are,

$$A^a = k\left(\frac{r+N}{r-N}\right)\delta^{a3}\sigma^3 = \frac{k}{2N}\left(\frac{r+N}{r-N}\right)\left[\frac{2(r^2 - N^2)}{2r^2 - 5Nr + 2N^2}\right]^{1/2}\delta^{a3}e^3,$$

$$F^a = -\frac{k}{(r-N)^2}\left[(e^0 \wedge e^3) + (e^1 \wedge e^2)\right]\delta^{a3} \tag{38}$$

for the solution to self-dual equation and

$$A^a = k\left(\frac{r-N}{r+N}\right)\delta^{a3}\sigma^3 = \frac{k}{2N}\left(\frac{r-N}{r+N}\right)\left[\frac{2(r^2-N^2)}{2r^2-5Nr+2N^2}\right]^{1/2}\delta^{a3}e^3,$$
$$F^a = \frac{k}{(r+N)^2}\left[(e^0 \wedge e^3) - (e^1 \wedge e^2)\right]\delta^{a3} \tag{39}$$

for the solution to anti-self-dual equation and where k is an arbitrary constant.

(iii) Abelianized YM instanton along the direction set by σ^1

Again, both the self-dual and anti-self-dual equations admit non-trivial solutions and they are

$$A^a = \frac{k}{(2r-N)^{1/4}(r-2N)}e^{-r/2N}\delta^{a1}\sigma^1 = \frac{k}{(2r-N)^{1/4}(r-2N)(r^2-N^2)^{1/2}}e^{-r/2N}\delta^{a1}e^1,$$
$$F^a = -\frac{k}{\sqrt{2}N}\frac{1}{(2r-N)^{3/4}(r-2N)^{3/2}}e^{-r/2N}\left[(e^0 \wedge e^1) + (e^2 \wedge e^3)\right]\delta^{a1} \tag{40}$$

for the solution to self-dual equation and

$$A^a = k(2r-N)^{1/4}(r-2N)e^{r/2N}\delta^{a1}\sigma^1 = k\frac{(2r-N)^{1/4}(r-2N)}{(r^2-N^2)^{1/2}}e^{r/2N}\delta^{a1}e^1,$$
$$F^a = \frac{k}{\sqrt{2}N}(2r-N)^{-1/4}(r-2N)^{1/2}e^{r/2N}\left[(e^0 \wedge e^1) - (e^2 \wedge e^3)\right]\delta^{a1} \tag{41}$$

for the solution to anti-self-dual equation. Note, however, that this last solution is *unphysical* and hence should be discarded as it fails to represent a localized soliton configuration.

(5) YM instanton in the de Sitter metric on S^4 background

This de Sitter (on S^4) gravitational instanton solution corresponds to

$$c_r = \left[1 + \left(\frac{r}{2a}\right)^2\right]^{-1}, \quad c_1 = c_2 = c_3 = \frac{r}{2}\left[1 + \left(\frac{r}{2a}\right)^2\right]^{-1} \tag{42}$$

where a is the radius of S^4 and it is a solution to Euclidean Einstein equation $R_{\mu\nu} = 8\pi\Lambda g_{\mu\nu}$. Thus the radius a of S^4 is related to the inverse of $\sqrt{\Lambda}$ as $a = \sqrt{3/8\pi\Lambda}$. Like the FS metric we studied earlier, this de Sitter metric on S^4 is another *compact* gravitational instanton having no boundary and hence is everywhere regular. As is well-known, de Sitter space is a space of constant curvature and hence is conformally-flat. Thus this de Sitter metric on S^4 has vanishing Weyl tensor, $C_{\mu\nu\alpha\beta} = 0$. This point is already evident from the fact that $c_1 = c_2 = c_3$ which indicates that the r =constant slices of this de Sitter metric on S^4 geometry are *round* S^3's with isometry group $SO(4)$. Thus the relevant ansätz for YM gauge connection is simply

$$A^a = f(r)\sigma^a$$

for reasons stated earlier and the associated field strength and the (anti) self-dual equation read

$$\begin{aligned} F^a &= \frac{f'}{c_r c_a}(e^0 \wedge e^a) + \frac{1}{2}\epsilon^{abc}\frac{f(f-1)}{c_b c_c}(e^b \wedge e^c), \\ &\pm \frac{f'}{f(f-1)} = \left(\frac{c_r}{c_1}\right) \end{aligned} \tag{43}$$

where "+" for self-dual and "−" for anti-self-dual equations. Obviously. this is the special case when $g(r) = 0$ in the more general case in eq.(10) we have been discussing. Then the standard YM instanton solutions (the physical one) can be constructed in a quite straight-forward manner and they are

$$\begin{aligned} A^a &= \left[1 + \left(\frac{r}{2a}\right)^2\right]^{-1} \sigma^a = \frac{2}{r}e^a, \\ F^a &= -\frac{1}{a^2}[(e^0 \wedge e^a) + \frac{1}{2}\epsilon^{abc}(e^b \wedge e^c)] \end{aligned} \tag{44}$$

for the solution to self-dual equation and

$$\begin{aligned} A^a &= \left[1 + \left(\frac{2a}{r}\right)^2\right]^{-1} \sigma^a = \frac{r}{2a^2}e^a, \\ F^a &= \frac{(4a)^2}{r^4}[(e^0 \wedge e^a) - \frac{1}{2}\epsilon^{abc}(e^b \wedge e^c)] \end{aligned} \tag{45}$$

for the solution to anti-self-dual equation. Note that these solutions in de Sitter (on S^4) instanton background are legitimate instanton solutions, namely one needs not worry about the seemingly divergent field energy upon integration over all space since the background de Sitter metric is a "compact" instanton with finite proper volume.

IV. Analysis of the nature of solutions to (anti) self-dual YM equation

We now would like to examine the nature of the solutions to (anti) self-dual YM equation in the background of various GI discussed in the previous section. Among other things, an interesting lesson we learned from this study is that, although expected to some extent, the chances for the existence of standard YM instanton solutions (to (anti) self-dual equations) get smaller as the degree of isometry owned by each gravitational instanton gets lower from, say, the de Sitter GI to the ones with self-dual Riemann or Weyl tensor and then next to the ones without. Next, concerning the discovered structure of the $SU(2)$ YN instanton solutions supported by these typical GI, there appears to be an interesting point worthy of note. First, recall that the relevant ansắtz for the YM gauge connection is of the form $A^a = f(r)\sigma^a$ in the highly symmetric de Sitter instanton background with topology of $R \times (\text{round})S^3$ and of the form $A^a = f(r)\sigma^a + g(r)\delta^{a3}\sigma^3$ in the less symmetric GI backgrounds with topology of $R \times (\text{squashed})S^3$. Here, however, the physical interpretation of the nature of YM gauge potential solutions A^a is rather unclear when they are expressed in terms of the left-invariant 1-forms $\{\sigma^a\}$ or the orthonormal basis e^A in eq.(6). Thus in order to get a better insight into the physical meaning of the structure of these YM connection ansắtz, we now try to re-express the left-invariant 1-forms $\{\sigma^a\}$ forming a basis on S^3 in terms of more familiar Cartesian coordinate basis. And this can be achieved by first relating the polar coordinates $(r,\ \theta,\ \phi,\ \psi)$ to Cartesian (t, x, y, z) coordinates (note, here, that t is not the usual "time" but just another spacelike coordinate) given by [5]

$$x + iy = r\cos\frac{\theta}{2}\exp[\frac{i}{2}(\psi+\phi)], \quad z + it = r\sin\frac{\theta}{2}\exp[\frac{i}{2}(\psi-\phi)], \tag{46}$$

where $x^2 + y^2 + z^2 + t^2 = r^2$ which is the equation for S^3 with radius r. From this coordinate transformation law, one now can relate the non-coordinate basis to the Caetesian coordinate basis as

$$\begin{pmatrix} dr \\ r\sigma_x \\ r\sigma_y \\ r\sigma_z \end{pmatrix} = \frac{1}{r}\begin{pmatrix} x & y & z & t \\ -t & -z & y & x \\ z & -t & -x & y \\ -y & x & -t & z \end{pmatrix}\begin{pmatrix} dx \\ dy \\ dz \\ dt \end{pmatrix} \tag{47}$$

where $\{\sigma_x = -\sigma^1/2,\ \sigma_y = -\sigma^2/2,\ \sigma_z = -\sigma^3/2\}$. Still, however, the meaning of YM gauge connection ansätz rewritten in terms of the Cartesian coordinate basis $dx^\mu = (dt,\ dx,\ dy,\ dz)$ as above does not look so apparent. Thus we next introduce the so-called *'tHooft tensor* [1,11] defined by

$$\eta^{a\mu\nu} = -\eta^{a\nu\mu} = (\epsilon^{0a\mu\nu} + \frac{1}{2}\epsilon^{abc}\epsilon^{bc\mu\nu}). \tag{48}$$

Then the left-invariant 1-forms can be cast to a more concise form $\sigma^a = 2\eta^a_{\mu\nu}(x^\nu/r^2)dx^\mu$. Therefore, the YM instanton solution, in Cartesian coordinate basis, can be written as

$$A^a = A^a_\mu dx^\mu = 2\left[f(r) + g(r)\delta^{a3}\right]\eta^a_{\mu\nu}\frac{x^\nu}{r^2}dx^\mu \tag{49}$$

in the background of TN, EH, FS, and TB GI with topology of $R \times (\text{squashed})S^3$. Now in order to appreciate the meaning of this structure, we go back to the flat space situation. As is well-known, the standard BPST [1] SU(2) YM instanton solution in flat space takes the form $A^a_\mu = 2\eta^a_{\mu\nu}[x^\nu/(r^2+\lambda^2)]$ with λ being the size of the instanton. Recall, however, that separately from this BPST instanton solution, there is another non-trivial solution to the YM field equation of the form $A^a_\mu = \eta^a_{\mu\nu}(x^\nu/r^2)$ found long ago by De Alfaro, Fubini, and Furlan [10]. (Note that the pure gauge is given by $A^a_\mu = 2\eta^a_{\mu\nu}(x^\nu/r^2)$. Thus the ordinary instanton solution interpolates between the trivial vacuum $A^a_\mu = 0$ at $r = 0$ and another vacuum represented by the pure gauge above at $r \to \infty$ and the meron solution can be thought of as a "half a vacuum gauge".) This second solution is called "meron" [11] as it carries a half unit of topological charge and is known to play a certain role concerning the quark confinement [11]. It, however, exhibits singularity at its center $r = 0$ and hence has a diverging action and falls like $1/r$ as $r \to \infty$. Thus we are led to the conclusion that the YM instanton solution in typical GI backgrounds possess the structure of (curved space version of) meron at large r. As is well-known, in flat spacetime meron does not solve the 1st order (anti) self-dual equation although it does the second order YM field equation. Thus in this sense, this result seems remarkable since it implies that in the GI backgrounds, the (anti) self-dual YM equation admits solutions which exhibit the configuration of meron solution at large r in contrast to the flat spacetime case. And we only conjecture that when passing from the flat (R^4) to GI ($R \times S^3$) geometry, the closure of the topology of part of the manifold appears to turn the structure of the instanton solution from that of standard BPST into that of meron. The concrete form of the YM instanton solutions in each of these GI backgrounds written in terms of Cartesian coordinate basis as in eq.(49) will be given below after we comment on one more thing.

Finally, we turn to investigation of other physical quantities such as the topological charge of each of these solutions and the estimate of the instanton contributions to the intervacua tunnelling amplitude which can serve as crucial indicators in determining the true physical natures of these solutions. It has been pointed out in the literature that both in the background of Euclidean Schwarzschild geometry [2] and in the Euclidean de Sitter space [3], the (anti) instanton solutions have the Pontryagin index of $\nu[A] = \pm 1$ and hence give the contribution to the (saddle point approximation to) intervacua tunnelling amplitude of $\exp\left[-8\pi^2/g_c^2\right]$, which, interestingly, are the same as their flat space counterparts even though these curved space YM instanton solutions do not correspond to gauge transformations of any flat space instanton solution [1]. This unexpected and hence rather curious property, however, turns out not to persist in YM instantons in these GI backgrounds we studied here. In order to see this, we begin with the careful definition of the Pontryagin index or second Chern class in the presence of the non-trivial background geometry of GI.

Consider that we would like to find an index theorem for the manifold (M) with boundary (∂M). Namely, we now need an extended version of index theorem with boundary. To this question, an appropriate answer has been provided by Atiyah, Patodi, and Singer (APS) [12]. According to their extended version of index theorem, the total index, say, of a given geometry and of a gauge field receives contributions, in addition to that from the usual bulk term ($V(M)$), from a local boundary term ($S(\partial M)$) and from a non-local boundary term ($\xi(\partial M)$). The bulk term is the usual term appearing in the ordinary index theorem without boundary and involves the integral over M of terms quadratic in curvature tensor of the geometry and in field strength tensor of the gauge field. The local boundary term is given by the integral over ∂M of the Chern-Simons forms for both the geometry and the gauge field while the non-local boundary term is given by a constant times the "APS η-invariant" [5] of the boundary. And this last non-local boundary term becomes relevant and meaningful when Dirac spinor field is present and interacts with the geometry and the gauge field. Now specializing to the case at hand in which we are interested in the evaluation of the instanton number or the second Chern class of the YM gauge field *alone*, we only need to pick up the terms in the gauge sector in this APS index theorem which reads [5]

$$\nu[A] = Ch_2(F) = \frac{-1}{8\pi^2}[\int_{M=R\times S^3} tr(F\wedge F) - \int_{\partial M=S^3} tr(\alpha\wedge F)|_{r=r_0}] \tag{50}$$

where $\alpha \equiv (A - A')$ is the "second fundamental form" *at* the boundary $r = r_0$ and by definition [5] A' has only *tangential* components on the boundary $\partial M = S^3$. Recall, however, that our choice of ansätz for the YM gauge connection involves the gauge fixing $A_r = 0$ as we mentioned earlier. Namely, both A and A' possess only tangential components (with respect to the $r = r_0$ boundary) at any $r = r_0$ and hence $\alpha \equiv (A - A') = 0$ identically there. As a result, even in the presence of the boundaries, the terms in the YM gauge sector in the APS index theorem remain the same as in the case of index theorem with no boundary, namely, only the bulk term survives in eq.(50) above. Thus what remains is just a straightforward computation of this bulk term and it becomes easier when performed in terms of orthonormal basis $e^A = \{e^0 = c_r dr, \ \ e^a = c_a\sigma^a\}$, in which case,

$$\begin{aligned} tr(F\wedge F) &= \frac{1}{2}(F^a\wedge F^a) = \frac{1}{2}(\frac{1}{4})\epsilon_{ABCD}F^a_{AB}F^a_{CD}\sqrt{g}d^4x \\ &= (F^1_{01}F^1_{23} + F^2_{02}F^2_{31} + F^3_{03}F^3_{12})\sqrt{g}d^4x, \\ \int_{M=R\times S^3} d^4x\sqrt{g} &= \int_R dr(c_r c_1 c_2 c_3)\int_0^{4\pi} d\psi \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \\ &= 16\pi^2 \int_R dr(c_r c_1 c_2 c_3) \end{aligned} \tag{51}$$

where we used $\sqrt{g} = |dete| = c_r c_1 c_2 c_3 \sin\theta$. The period for the $U(1)$ fibre coordinate ψ for the EH metric, however, is 2π rather than 4π to remove the bolt singularity at $r = a$ as we mentioned earlier. This completes the description of the method for computing the topological charge of each solution. Our next job, then, is the estimate of the instanton contributions to the intervacua tunnelling amplitudes. Generally, the saddle point approximation to the intervacua tunnelling amplitude is given by

$$\Gamma_{GI} \sim \exp\left[-I_{GI}(instanton)\right] \tag{52}$$

where the subscript "GI" denotes corresponding quantities in the GI backgrounds and $I_{GI}(instanton)$ represents the Euclidean YM theory action evaluated at the YM instanton solution, i.e.,

$$I_{GI}(instanton) = \int_{R\times S^3} d^4x\sqrt{g}\left[\frac{1}{4g_c^2}F^a_{\mu\nu}F^{a\mu\nu}\right] = \left(\frac{8\pi^2}{g_c^2}\right)|\nu[A]| \tag{53}$$

where we used $4tr(F\wedge F) = F^a_{\mu\nu}\tilde{F}^{a\mu\nu}\sqrt{g}d^4x$ and the (anti)self-duality relation $F^a = \pm\tilde{F}^a$. The calculation of the Pontryagin indices and hence the Euclidean YM actions we just described is indeed quite straightforward.

In the following, as we promised, we now provide the expression for the YM instanton solutions in each of these GI backgrounds written in terms of Cartesian coordinate basis to study its structure one by one in detail and also we demonstrate the explicit evaluation of the topological charge values and the estimate of the contributions to the intervacua tunnelling amplitude in order eventually to determine the physical nature of each solution.

(1) YM instanton in Taub-NUT metric background

In terms of the ansätz functions $f(r)$ and $g(r)$ for the YM gauge connection in GI backgrounds given in eq.(10), the standard instanton solutions in TN metric amount to

$$\begin{aligned} f(r) &= \left(\frac{r-m}{r+m}\right), \quad g(r) = \left(\frac{2m}{r+m}\right)\left(\frac{r-m}{r+m}\right), \\ f(r) &= -\left(\frac{r-m}{r+m}\right), \quad g(r) = 2\left(\frac{r+2m}{r+m}\right)\left(\frac{r-m}{r+m}\right) \end{aligned} \tag{54}$$

for self-dual and anti-self-dual YM equations respectively. Therefore, when expressed in Cartesian coordinate basis as in eq.(49), the solutions take the forms

$$A^a_\mu = 2\left(\frac{r-m}{r+m}\right)\left[1+\left(\frac{2m}{r+m}\right)\delta^{a3}\right]\eta^a_{\mu\nu}\frac{x^\nu}{r^2}, \quad (55)$$
$$A^a_\mu = 2\left(\frac{r-m}{r+m}\right)\left[-1+2\left(\frac{r+2m}{r+m}\right)\delta^{a3}\right]\eta^a_{\mu\nu}\frac{x^\nu}{r^2}$$

for self-dual and anti-self-dual case respectively. Some comments regarding the features of these solutions are now in order. i) They appear to be singular at the center $r = 0$ but it should not be a problem as $r \geq m$ for the background TN metric and hence the point $r = 0$ is absent. ii) It is interesting to note that the solutions become vacuum gauge $A^a_\mu = 0$ at the boundary $r = m$ which has the topology of S^3. iii) For $r \to \infty$, the solutions asymptote to another vacuum gauge $|A^a_\mu| = 2\eta^a_{\mu\nu}(x^\nu/r^2)$.

We now turn to the computation of the topological charge, i.e., the Pontryagin index of these YM solution. The relevant quantities involved in this computation are the ones in eq.(51) and they, for the case at hand, are

$$(c_r c_1 c_2 c_3) = \frac{m}{8}(r^2 - m^2), \quad (56)$$
$$F^a_{\mu\nu}\tilde{F}^{a\mu\nu} = 4(F^1_{01}F^1_{23} + F^2_{02}F^2_{31} + F^3_{03}F^3_{12}) = -24\frac{(8m)^2}{(r+m)^6}.$$

Thus we have

$$\nu[A] = \left(\frac{-1}{32\pi^2}\right)16\pi^2\int_m^\infty dr\frac{m}{8}(r^2-m^2)\left[-24\frac{(8m)^2}{(r+m)^6}\right]$$
$$= 1. \quad (57)$$

Then next the Euclidean YM action evaluated at these instanton solutions and hence the saddle point approximation to the intervacua tunnelling amplitude are given respectively by

$$I_{GI}(instanton) = \left(\frac{8\pi^2}{g_c^2}\right)|\nu[A]| = \frac{8\pi^2}{g_c^2}, \quad (58)$$
$$\Gamma_{GI} \sim \exp\left[-I_{GI}(instanton)\right] = \exp\left(-8\pi^2/g_c^2\right).$$

(2) YM instanton in Eguchi-Hanson metric background

The standard instanton solutions in EH metric amount to

$$f(r) = \left[1-\left(\frac{a}{r}\right)^4\right]^{1/2}, \quad g(r) = \left[1+\left(\frac{a}{r}\right)^4\right] - \left[1-\left(\frac{a}{r}\right)^4\right]^{1/2}, \quad (59)$$
$$f(r) = -\left[1-\left(\frac{a}{r}\right)^4\right]^{1/2}, \quad g(r) = \left[1+\left(\frac{a}{r}\right)^4\right] + \left[1-\left(\frac{a}{r}\right)^4\right]^{1/2}$$

for self-dual and anti-self-dual YM equations respectively. Thus in Cartesian coordinate basis, the solutions take the forms

$$A^a_\mu = 2\left\{\left[1-\left(\frac{a}{r}\right)^4\right]^{1/2} + \left(\left[1+\left(\frac{a}{r}\right)^4\right] - \left[1-\left(\frac{a}{r}\right)^4\right]^{1/2}\right)\delta^{a3}\right\}\eta^a_{\mu\nu}\frac{x^\nu}{r^2}, \tag{60}$$

$$A^a_\mu = 2\left\{-\left[1-\left(\frac{a}{r}\right)^4\right]^{1/2} + \left(\left[1+\left(\frac{a}{r}\right)^4\right] + \left[1-\left(\frac{a}{r}\right)^4\right]^{1/2}\right)\delta^{a3}\right\}\eta^a_{\mu\nu}\frac{x^\nu}{r^2}$$

for self-dual and anti-self-dual cases respectively. Some comments regarding the features of these solutions are now in order. i) Again, they appear to be singular at the center $r = 0$ but it should not be a problem as $r \geq a$ for the background EH metric and hence the point $r = 0$ is absent. ii) The solutions become $A^a_\mu = 4\eta^a_{\mu\nu}\delta^{a3}(x^\nu/r^2)$ at the boundary $r = a$ which has the topology of S^3/Z_2. iii) For $r \to \infty$, the solutions asymptote to the vacuum gauge $|A^a_\mu| = 2\eta^a_{\mu\nu}(x^\nu/r^2)$.

We turn now to the computation of the Pontryagin index of these YM solution. For the case at hand, the relevant quantities involved in this computation are

$$(c_r c_1 c_2 c_3) = \frac{1}{8}r^3, \quad F^a_{\mu\nu}\tilde{F}^{a\mu\nu} = 24\left(\frac{4a^4}{r^6}\right)^2. \tag{61}$$

Thus we have

$$\nu[A] = \left(\frac{-1}{32\pi^2}\right)8\pi^2\int_a^\infty dr\frac{1}{8}r^3\left[24\left(\frac{4a^4}{r^6}\right)^2\right] = -\frac{3}{2} \tag{62}$$

where we set the range for the $U(1)$ fibre coordinate as $0 \leq \psi \leq 2\pi$ rather than $0 \leq \psi \leq 4\pi$ for the reason stated earlier. Note particularly that it is precisely this point that renders the Pontryagin index of this solution *fractional* because otherwise, it would come out as -3 instead. Then next the Euclidean YM action evaluated at these instanton solutions and hence the saddle point approximation to the intervacua tunnelling amplitude are given respectively by

$$I_{GI}(instanton) = \left(\frac{8\pi^2}{g_c^2}\right)|\nu[A]| = \frac{12\pi^2}{g_c^2}, \tag{63}$$

$$\Gamma_{GI} \sim \exp\left[-I_{GI}(instanton)\right] = \exp\left(-12\pi^2/g_c^2\right).$$

(3) YM instanton in Fubini-Study metric on CP^2 background

The standard instanton solutions in FS metric amount to

$$f(r) = \left[1+\frac{1}{6}\Lambda r^2\right]^{-1/2}, \quad g(r) = \frac{[1+\Lambda r^2/12]}{[1+\Lambda r^2/6]} - \frac{1}{[1+\Lambda r^2/6]^{1/2}}, \tag{64}$$
$$f(r) = -\left[1+\frac{1}{6}\Lambda r^2\right]^{-1/2}, \quad g(r) = \frac{[1+\Lambda r^2/12]}{[1+\Lambda r^2/6]} + \frac{1}{[1+\Lambda r^2/6]^{1/2}}$$

for self-dual and anti-self-dual YM equations respectively. Then in terms of Cartesian coordinate basis, the solutions take the forms

$$A^a_\mu = \frac{2}{[1+\Lambda r^2/6]^{1/2}}\left\{1+\left[\frac{(1+\Lambda r^2/12)}{(1+\Lambda r^2/6)^{1/2}}-1\right]\delta^{a3}\right\}\eta^a_{\mu\nu}\frac{x^\nu}{r^2}, \tag{65}$$
$$A^a_\mu = \frac{2}{[1+\Lambda r^2/6]^{1/2}}\left\{-1+\left[\frac{(1+\Lambda r^2/12)}{(1+\Lambda r^2/6)^{1/2}}+1\right]\delta^{a3}\right\}\eta^a_{\mu\nu}\frac{x^\nu}{r^2}$$

for self-dual and anti-self-dual cases respectively. Now, note that : i) The solution to the self-dual YM equation $A^a_\mu = 2\eta^a_{\mu\nu}(x^\nu/r^2)$ looks singular at the center $r=0$ since $0 \le r < \infty$ for the background FS metric and hence the point $r=0$ is present. But this is just a pure gauge representing a vacuum and thus should not be a trouble. Next, the solution to the anti-self-dual YM equation, $|A^{1,2}_\mu| = -2\eta^{1,2}_{\mu\nu}(x^\nu/r^2)$ and $|A^3_\mu| = 2\eta^3_{\mu\nu}(x^\nu/r^2)$ is again a pure gauge having a vanishing field strength. ii) For $r \to \infty$, the solutions asymptote to $|A^{1,2}_\mu| = 2\sqrt{6/\Lambda}\eta^{1,2}_{\mu\nu}(x^\nu/r^3) \to 0$ and $|A^3_\mu| = \eta^3_{\mu\nu}(x^\nu/r^2)$ which is a component of flat space meron solution.

Turning now to the calculation of the Pontryagin index of these YM solutions, again the relevant quantities involved in this computation are

$$(c_r c_1 c_2 c_3) = \frac{r^3}{8(1+\frac{1}{6}\Lambda r^2)^3}, \quad F^a_{\mu\nu}\tilde{F}^{a\mu\nu} = \frac{4}{3}\Lambda^2.$$

Thus

$$\nu[A] = \left(\frac{-1}{32\pi^2}\right)16\pi^2\int_0^\infty dr\frac{r^3}{8(1+\frac{1}{6}\Lambda r^2)^3}\left[\frac{4}{3}\Lambda^2\right] = -\frac{3}{4}. \tag{66}$$

Next the Euclidean YM action and the saddle point approximation to the intervacua tunnelling amplitude are given respectively by

$$I_{GI}(instanton) = \left(\frac{8\pi^2}{g_c^2}\right)|\nu[A]| = \frac{6\pi^2}{g_c^2}, \quad (67)$$

$$\Gamma_{GI} \sim \exp\left[-I_{GI}(instanton)\right] = \exp\left(-6\pi^2/g_c^2\right).$$

(4) YM instanton in the de Sitter metric on S^4 background

In terms of the ansätz functions $f(r)$ for the YM gauge connection given earlier as $A^a = f(r)\sigma^a$, the standard instanton solutions in de Sitter metric amount to

$$f(r) = \left[1 + \left(\frac{r}{2a}\right)^2\right]^{-1}, \quad f(r) = \left[1 + \left(\frac{2a}{r}\right)^2\right]^{-1} \quad (68)$$

for self-dual and anti-self-dual YM equations respectively. Then in terms of Cartesian coordinate basis, the solutions take the forms

$$A^a_\mu = \frac{2}{[1 + (r/2a)^2]}\eta^a_{\mu\nu}\frac{x^\nu}{r^2}, \quad A^a_\mu = \frac{2}{[1 + (2a/r)^2]}\eta^a_{\mu\nu}\frac{x^\nu}{r^2} \quad (69)$$

for self-dual and anti-self-dual cases respectively. Now, note that : i) The solution to the self-dual YM equation $A^a_\mu = 2\eta^a_{\mu\nu}(x^\nu/r^2)$ is again a pure gauge representing a vacuum and thus should not be a trouble although the point $r = 0$ is present in the background de Sitter space. The solution to the anti-self-dual YM equation also approaches the vacuum, i.e., $A^a_\mu \simeq (1/2a^2)\eta^a_{\mu\nu}x^\nu \to 0$. ii) For $r \to \infty$, the solutions asymptote to $A^a_\mu = 8a^2\eta^a_{\mu\nu}(x^\nu/r^4) \sim 0$ for the self-dual case and $A^a_\mu = 2\eta^a_{\mu\nu}(x^\nu/r^2)$ which is the pure gauge for the anti-self-dual case.

Lastly, turning to the calculation of the Pontryagin index of these YM solutions, we first obtain the relevant quantities involved in this computation which are

$$(c_r c_1 c_2 c_3) = \frac{r^3}{8[1 + (r/2a)^2]^4}, \quad (70)$$

$$F^a_{\mu\nu}\tilde{F}^{a\mu\nu} = \frac{1}{2}\epsilon_{ABCD}F^a_{AB}F^a_{CD} = \frac{12}{a^4}.$$

Thus

$$\nu[A] = \left(\frac{-1}{32\pi^2}\right)16\pi^2\int_0^\infty dr\frac{r^3}{8[1 + (r/2a)^2]^4}\left[\frac{12}{a^4}\right] = -1. \quad (71)$$

Then the Euclidean YM action and the saddle point approximation to the intervacua tunnelling amplitude are given respectively by

$$I_{GI}(instanton) = \left(\frac{8\pi^2}{g_c^2}\right)|\nu[A]| = \frac{8\pi^2}{g_c^2}, \tag{72}$$
$$\Gamma_{GI} \sim \exp\left[-I_{GI}(instanton)\right] = \exp\left(-8\pi^2/g_c^2\right).$$

Let us now discuss the behavior of these solutions as $r \to 0$ once again to stress that they really do not exhibit singular behaviors there. For TN, EH, and TB instantons, the ranges for radial coordinates are $m \leq r < \infty$, $a \leq r < \infty$, and $2N \leq r < \infty$ respectively. Since the point $r = 0$ is absent in these manifolds, the solutions in these GI are everywhere regular. For the rest of the "compact" gravitational instantons, i.e., FS on CP^2 and de Sitter on S^4, however, the radial coordinate runs $0 \leq r < \infty$. Thus the point $r = 0$ indeed is present in these compact instantons. The solutions in FS and de Sitter backgrounds, however, seem to have no trouble either as they are essentially vacuum gauges having vanishing field strength there at $r = 0$. At large r, on the other hand, all the solutions appear to take the structure close to that of meron solution in flat space. Another interesting point worthy of note is that the solutions in TN and de Sitter backgrounds exhibit a generic property of the instanton solution in that they do interpolate a vacuum at $r = m$ ($r = 0$) and another vacuum at $r \to \infty$. Namely, the solutions in these GI backgrounds appear to exhibit features of both meron such as their large r behavior and instanton such as interpolating configurations between two vacua in some cases. Next, we analyze the meaning of the topological charge values of the solutions and their contributions to the intervacua tunnelling amplitudes. Except for the solutions in the background of TN metric and de Sitter metric, generally the solutions in other GI backgrounds such as EH and FS carry *fractional* topological charges smaller or greater than unity in magnitude. Here, however, the solution in EH metric background carries the half-integer Pontryagin index actually because the range for the $U(1)$ fibre coordinate is $0 \leq \psi \leq 2\pi$ and hence the boundary of EH space is S^3/Z_2. For the FS (on CP^2) case, it is unclear what is the true origin for the fractional Pontryagin index. Therefore the fact that solutions in GI backgrounds generally carry fractional topological charges appears to be another manifestation for mixed instanton-meron nature of the solutions. Thus to summarize, the solutions in TN and de

Sitter backgrounds particularly display features generic in the standard instanton while in the case of those in EH and FS backgrounds, such generic features of the instanton is somewhat obscured by meron-type natures. There. however, is one obvious consensus. All the solutions in these GI backgrounds are non-singular at their centers and have finite Euclidean YM action. And this last point allows us to suspect that these solutions are more like instantons in their generic nature although looks rather like merons in their structures.

V. Concluding remarks

We now summarize the results and close with some comments. As we stressed earlier in the introduction, when it comes to the topological aspect, gravity may have marked effects even at the level of elementary particle physics despite its negligibly small relative strength well below the Planck scale. Although this intriguing possibility has been pointed out long ago, surprisingly little attempt has been made toward the demonstration of this phenomenon in relevant physical systems. Thus in the present work, we took a concrete step toward in this direction. Namely, we attempted to construct in an explicit and precise manner the $SU(2)$ YM instanton solutions practically in all known gravitational instanton backgrounds. And in doing so, the task of solving coupled Einstein-Yang-Mills equations for the metric and YM gauge field has been greatly simplified by the fact that in Euclidean signature, the YM field does not disturb the geometry as its energy-momentum tensor vanishes identically as long as one looks only for the YM instanton solutions having (anti) self-dual field strength. Among other things, an interesting lesson we learned from this study is that, although expected to some extent, the chances for the existence of standard YM instanton solutions (to (anti) self-dual equations) get smaller as the degree of isometry owned by each gravitational instanton gets lower from, say, the de Sitter GI to the ones with self-dual Riemann or Weyl tensor and then next to the ones without. As demonstrated, it is also intersting to note that the solutions turn out to take the structure of merons at large r and generally carry fractional topological charge values. Nevertheless, it seems more appropriate to conclude that the solutions still should be identified with (curved space version of) instantons as they are solutions to 1st order (anti) self-dual equation and are everywhere regular having finite YM action. However, these curious mixed characteristics of the solutions to (anti) self-dual YM equation in GI backgrounds appear to invite us to take them more seriously and further explore potentially interesting physics associated with them.

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